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A Feynman–Kac formula for anticommuting Brownian motion

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Abstract

Motivated by application to quantum physics, anticommuting analogues of Wiener measure and Brownian motion are constructed. The corresponding Itô integrals are defined and the existence and uniqueness of solutions to a class of stochastic differential equations is established. This machinery is used to provide a Feynman–Kac formula for a class of Hamiltonians. Several specific examples are considered.

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1. Introduction

Anticommuting variables occur in physics when either a supersymmetry or a BRST symmetry occurs. In the first place such variables occur as the parameters of each of these two kinds of symmetry transformation, but they also occur when the operators of the quantized theory are represented by differential operators on function spaces: the presence of canonical *anti*commutation relations means that the functions involved are functions of anticommuting variables, an idea which goes back originally to work of Martin [1] and ideas of Schwinger [2], and was extensively developed by Berezin [3] and by De Witt [4]. Anticommuting variables are not used to model physical quantities directly; their use is motivated by the algebraic properties of the function spaces of these variables. In application to physics, results which are real or complex numbers emerge after what has become known as Berezin integration (defined by equation (5) in section 2), which essentially takes a trace. The approach using anticommuting variables is particularly useful in the context of supersymmetry and BRST symmetry because Bose and Fermi (or ghost) degrees of freedom, which are related by symmetry transformations, are both handled in the same way.

Path integral quantization in this approach has been developed in terms of limits of time-slicing by a number of authors, starting from the work of Martin [1] with further work by, among others, Marinov [5]. A clear account of this use of Grassmann variables in fermionic quantization is given by Swanson [6].

In this paper we investigate a more rigorous, mathematical approach to the path integral quantization of ghost Hamiltonians by developing anticommuting analogues to various

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constructions in probability theory (such as Brownian motion and stochastic calculus) and applying these objects to establish a Feynman–Kac formula for a wide family of ghost Hamiltonians of the kind which occur when quantizing systems in the BRST approach. These anticommuting analogues are constructed in close parallel to their classical commuting counterparts, so that the two may readily be combined to give a 'super' theory in a geometric setting. The anticommuting Brownian motion developed here is distinct from that developed by one of the authors for fermionic quantization [7, 8], essentially because these two classes of theory have distinct free Hamiltonians.

Other approaches to quantization of fermionic and ghost degrees of freedom have been considered by several authors: it is not possible to give a full list, but examples are the work of Gaveau and Schulman [9], Applebaum and Hudson [10] and Hudson and Lindsay [11], and Kupsch [12]. Closest to the paper presented here is the work of Barnett *et al* [13, 14] and of Streater and Hasagawa [15], as will be discussed in more detail in section 4.

Although, as remarked above, the anticommuting variables used in this paper do not directly model physical quantities, they do provide a 'classical' framework for fermions and ghosts, which have meaning only at the quantum level, and might find application in other contexts, such as the more exotic diffusions of Metzler *et al* [16].

2. Anticommuting variables

In this section we briefly describe the space of anticommuting variables from which our processes are built, together with the key features of the analysis of functions of such variables. Further details may be found in [8]. The approach taken, using Grassmann algebras, is more concrete and more particular than strictly necessary; a more abstract approach is possible, which would be more mathematically economical and elegant, but would not relate in so direct a way to the standard methods of stochastic calculus.

The basic anticommuting algebra used is the real Grassmann algebra with an infinite number of generators; this algebra, which is denoted $\mathbb{R}_S$, is a superalgebra with $\mathbb{R}_S := \mathbb{R}_{S,0} \oplus \mathbb{R}_{S,1}$ where $\mathbb{R}_{S,0}$ is the even part, consisting of elements which are a linear combination of terms each containing a product of even numbers of the anticommuting generators, while $\mathbb{R}_{S,1}$ is the odd part. We will normally consider homogeneous elements, that is elements A which are either even or odd, with parity denoted by ϵ_A so that $\epsilon_A = i$ if A is in $\mathbb{R}_{S,i}$, $i = 0, 1$. The algebra $\mathbb{R}_S$ is supercommutative, that is $AB = (-1)^{\epsilon_A \epsilon_B} BA$, so that in particular $\alpha\beta = -\beta\alpha$ if and only if both α and β are odd. We shall not need to be concerned with analysis on this space directly, and so do not need to specify any norm. Our use of the space will be purely algebraic.

The functions with which we shall principally be concerned, because of their rôle in ghost quantization, have as domain the space $\mathbb{R}_S^{0,m} := (\mathbb{R}_{S,1})^m$. A typical element of this space is $\eta := (\eta^1, \ldots, \eta^m)$. (It will be assumed that m is an even number in this paper, although in other contexts this is not necessarily the case.) We will consider functions on this space which are *supersmooth* [4, 17], that is (in this simple context where we consider purely anticommuting variables) multinomials in the anticommuting variables. These may be written in a standard form if we introduce multi-index notation: let M_n denote the set of all multi-indices of the form $\mu := \mu_1 \ldots \mu_k$ with $1 \leqslant \mu_1 < \cdots < \mu_k \leqslant m$ together with the empty multi-index $\emptyset$; also let $|\mu|$ denote the length of the multi-index μ, $\eta^\emptyset := 1$ (the unit of $\mathbb{R}_S$) and $\eta^\mu := 1\eta^{\mu_1} \ldots \eta^{\mu_{|\mu|}}$. A supersmooth function is then a function F of the form

$$F : \mathbb{R}_S^{0,m} \longrightarrow \mathbb{R}_S \qquad (\eta^1, \ldots, \eta^m) \mapsto \sum_{\mu \in M_m} F_\mu \eta^\mu \tag{1}$$

where the coefficients F_μ are real or complex numbers.

Differentiation of multinomial functions of anticommuting variables is defined by linearity together with the rule

$$\frac{\partial\eta^\mu}{\partial\eta^j} = \begin{cases} (-1)^{\ell-1}\eta^{\mu_1}\cdots\widehat{\eta^\ell}\cdots\eta^{\mu_{|\mu|}} & \text{if} \quad j=\mu_\ell \quad \text{for some} \quad \ell, 1 \leqslant \ell \leqslant |\mu| \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

where $\widehat{\ }$ indicates an omitted factor.

Functions of anticommuting variables obey the following Taylor theorem, which can be proved as in the classical case.

Theorem 2.1. *If F is a supersmooth function on $\mathbb{R}_S^{0,m}$ and ξ, η are elements of $\mathbb{R}_S^{0,m}$,*

$$\begin{aligned} F(\xi+\eta) - F(\xi) &= \eta^{a_1}\partial_{a_1}F(\xi) + \frac{1}{2!}\eta^{a_2}\eta^{a_1}\partial_{a_1}\partial_{a_2}F(\xi) \\ &\quad + \cdots + \frac{1}{(n-1)!}\eta^{a_{n-1}}\cdots\eta^{a_1}\partial_{a_1}\cdots\partial_{a_{n-1}}F(\xi) \\ &\quad + \int_0^1 \frac{(1-t)^{n-1}}{(n-1)!}\eta^{a_n}\cdots\eta^{a_1}\partial_{a_1}\cdots\partial_{a_n}F(\xi+t\eta)\,\mathrm{d}t. \end{aligned} \tag{3}$$

(Here and later the summation convention for repeated indices is used.) If the number of terms n is greater than the number of anticommuting variables m this takes the simpler form

$$F(\xi+\eta) = \sum_{\mu\in M_n} \eta^\mu \partial_{\tilde\mu}F(\xi) \tag{4}$$

where $\partial_{\tilde\mu} = \partial_{\mu_{|\mu|}}\ldots\partial_{\mu_1}$.

Integration of functions of these anticommuting variables is defined algebraically by the Berezin rule:

$$\int_{\mathcal{B}} \mathrm{d}^m\eta\, F(\eta) = F_{1\ldots m} \tag{5}$$

where $F(\eta) = \sum_{\mu\in M_m} F_\mu\eta^\mu$ as in (1), so that $F_{1\ldots m}$ is the coefficient of the highest-order term.

The space of supersmooth functions of m anticommuting variables will be denoted $\mathcal{F}(m)$, and is a 2^m-dimensional vector space. A norm on this space is defined by

$$|F|_G = \sum_{\mu\in M_n} |F_\mu| \tag{6}$$

where again $F(\eta) = \sum_{\mu\in M_m} F_\mu\eta^\mu$ as in (1). This norm has the Banach algebra property

$$|FG|_G \leqslant |F|_G|G|_G. \tag{7}$$

Any linear operator K on this space has integral kernel taking $\mathbb{R}_S^{0,m}\times\mathbb{R}_S^{0,m}$ into $\mathbb{R}_S$ defined by

$$Kf(\theta) = \int_{\mathcal{B}} \mathrm{d}^m\theta\, K(\eta,\theta)f(\theta). \tag{8}$$

3. Anticommuting probability and stochastic processes

While the standard integral for functions of anticommuting variables, the Berezin integral defined in equation (5), has no measure-theoretic or 'limit of a sum' aspect, it can be used to build an anticommuting analogue of probability theory by taking the consistency conditions of the Kolmogorov extension theory as the defining properties, as has been carried out in [7, 8]. The key definition of anticommuting probability space is now given. A restricted form of the definition, sufficient for this paper, is used, with more details and generality available in the references cited.

Definition 3.1. *A* $(0, m)$-anticommuting probability space of weight w *consists of:*

(a) a finite closed interval $[0, T]$ *of the real line;*
(b) for each finite set $B = \{t_1, \ldots, t_r\}$ *with* $0 \leqslant t_1 < \cdots < t_r \leqslant T$*, a supersmooth function* F_B *on* $(\mathbb{R}_S^{0,m})^r$ *such that:*

(i)
$$\int_{\mathcal{B}} \mathrm{d}^m\theta_1 \cdots \mathrm{d}^m\theta_r \, F_B(\theta_1, \ldots, \theta_r) = w \tag{9}$$
(where $\theta_1, \ldots, \theta_r$ *are each elements of* $\mathbb{R}_S^{0,m}$*);*

(ii) if $B = \{t_1, \ldots, t_r\}$ *and* $B' = \{t_1, \ldots t_{r-1}\}$ *then*
$$\int_{\mathcal{B}} \mathrm{d}^m\theta_r \, F_B(\theta_1, \ldots, \theta_r) = F_{B'}(\theta_1, \ldots, \theta_{r-1}). \tag{10}$$

Such a space will be denoted $((\mathbb{R}_S^{0,m})^{[0,T]}, \{F_B\}, \mathrm{d}\mu)$.

(The conditions (9) and (10) are analogous to the consistency conditions for finite-dimensional distributions.)

We can now define the notion of random variable on this space; we cannot use conventional measure theory, but must instead build an explicit limiting process into the definition.

Definition 3.2. *A* $(0, k)$-dimensional anticommuting random variable
$$G^i := (G_r^i, B_r) \qquad i = 1, \ldots, k \tag{11}$$
for the anticommuting probability space $((\mathbb{R}_S^{0,m})^{[0,T]}, \{F_B\}, \mathrm{d}\mu)$ *consists of:*

(a) a sequence of defining sets $B_1, B_2, \ldots,$ *each a finite subset of* $[0, T]$*;*
(b) a sequence of supersmooth functions $G_r : (\mathbb{R}_S^{0,m})^{|B_r|} \to \mathbb{R}_S^{0,k}$, $r = 1, 2, \ldots$ *(with components* $G_r^i, i = 1, \ldots, k$*) such that for each* $i = 1, \ldots, k$ *and each multinomial function* H *of* k *variables the sequence*
$$I_r(H) = \int_{\mathcal{B}} \mathrm{d}^m\theta_1 \ldots d^m\theta_{|B_r|} \, F_{B_r}(\theta_1, \ldots, \theta_{|B_r|}) H(G_r(\theta_1, \ldots, \theta_{|B_r|})) \tag{12}$$
tends to a limit as r *tends to infinity. (Here* $|B_r|$ *denotes the number of elements in the set* B_r*.)*

The limit of $I_r(H)$ *is called the* (anticommuting) expectation value *of* $H(G^i)$*, and we write*
$$\mathbb{E}_G[H(G^i)] \equiv \int \mathrm{d}\mu \, H(G^i) := \lim_{r\to\infty} I_r(H). \tag{13}$$
The case where there exists some finite number M *such that* $B_q = B_M$ *for all* $q > M$ *is called a* finitely defined anticommuting random variable.

The definition of a stochastic process is analogous to the conventional one.

Definition 3.3. *Let* A *be an interval contained in* $[0, T]$*. Then a collection*
$$X := \{ X_t \mid t \in A \} \tag{14}$$
of $(0, k)$*-dimensional random variables on an anticommuting probability space* $((\mathbb{R}_S^{0,m})^{[0,T]}, \{F_B\}, \mathrm{d}\mu)$ *is said to be a* $(0, k)$-dimensional stochastic process *on the space* $((\mathbb{R}_S^{0,m})^{[0,T]}, \{F_B\}, \mathrm{d}\mu)$ *if for each finite subset* A_α *of* A *the collection* $X := \{ X_t \mid t \in A_\alpha \}$ *is an anticommuting random variable on this space.*

In this paper we shall be concerned with stochastic processes which are built from solutions of stochastic differential equations.

We end this section with some useful but rather technical definitions starting with a notion of equality of random variables.

Definition 3.4. *If (X^i) and (Y^i) are two $(0, k)$-dimensional random variables and*

$$\mathbb{E}_G[H(X)] = \mathbb{E}_G[H(Y)] \tag{15}$$

for all multinomial functions H of k variables, then we say they are μ-equal*. This is written*

$$X^i =_\mu Y^i. \tag{16}$$

The next definition defines convergence of a sequence of random variables.

Definition 3.5. *If X is a $(0, k)$-dimensional random variable, X_r, $r = 1, 2, \ldots$ a sequence of $(0, k)$-dimensional random variables and*

$$\lim_{r\to\infty} |\mathbb{E}_G[H(X_r) - H(X)]| = 0 \tag{17}$$

for each multinomial function H then we say that X_r μ-converges *to X. This will be denoted*

$$\mu - \lim_{r\to\infty} X_r = X. \tag{18}$$

While other kinds of equality and convergence can be defined, these forms are sufficient for the purposes of this paper since the Feynman–Kac formula is built from expectations of anticommuting random variables.

4. Anticommuting Brownian motion

The anticommuting Brownian motion process will now be constructed. We start by defining anticommuting Wiener space, using finite-dimensional marginal distributions built from the heat kernel of the 'free' Hamiltonian for functions of m anticommuting variables. Recalling that we are assuming that m is even, this Hamiltonian is

$$H_F := \frac{1}{2} \mathrm{e}^{ij} \frac{\partial}{\partial \eta^i} \frac{\partial}{\partial \eta^j} \tag{19}$$

where the $m \times m$ matrix e in block diagonal form is

$$e = \begin{pmatrix} \epsilon & & \\ & \ddots & \\ & & \epsilon \end{pmatrix} \tag{20}$$

with $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The heat kernel $\mathrm{e}^{-H_F t}(\eta, \eta')$ of this Hamiltonian is

$$p(\eta - \eta', t) := (\surd t)^m \exp\left(\frac{e_{ji}(\eta^i - \eta'^i)(\eta^j - \eta'^j)}{2t} \right) \tag{21}$$

as may be verified by observing that $p(\eta - \eta', t)$ satisfies the equation

$$\frac{\partial}{\partial t} p(\eta, \eta', t) = -H_F\, p(\eta, \eta', t) \tag{22}$$

and reduces to the Grassmann delta function $\delta(\eta - \eta') := \Pi_{i=1}^m (\eta^i - \eta'^i)$ when $t = 0$.

Anticommuting Brownian motion is now defined to be the anticommuting stochastic process constructed from this heat kernel in the following way.

Definition 4.1. *Anticommuting Wiener space of dimension $(0, m)$ (where m is even) on the time interval $[0, T]$ is the anticommuting probability space $((\mathbb{R}_S^{0,m})^{[0,T]}, \{F_B\}, \mathrm{d}\mu)$ with*

$$F_{\{t_1,\ldots,t_N\}}(\eta_1, \ldots, \eta_N) := p(\eta_1, t_1) p(\eta_2 - \eta_1, t_2 - t_1) \cdots p(\eta_N - \eta_{N-1}, t_N - t_{N-1}) \tag{23}$$

for each finite set $\{t_1, \ldots, t_N\}$ of real numbers for which $0 \leqslant t_1 < \cdots < t_N \leqslant T$.

It follows immediately from the semigroup property of the heat kernel $p(\eta, \eta', t)$ that the finite-dimensional marginal distributions $F_{t_1,\ldots,t_N}$ satisfy the necessary consistency condition contained in definition 3.1, and then by direct calculation that the weight of the space is 1.

We now define m-dimensional anticommuting Brownian motion to be the stochastic process β_t defined by this anticommuting probability space, so that for any supersmooth function H of mN anticommuting variables, where N is a positive integer,

$$\begin{aligned}\mathbb{E}_G[H(\beta^a_{t_1}, \ldots, \beta^a_{t_N})] = \int_{\mathcal{B}} \mathrm{d}^m\theta_1 \cdots \mathrm{d}^m\theta_N\, p(\theta_1, t_1) p(\theta_2 - \theta_1, t_2 - t_1) \times \cdots \\ \times p(\theta_N - \theta_{N-1}, t_N - t_{N-1}) H(\theta_1, \ldots, \theta_N).\end{aligned} \tag{24}$$

The following expectations, which will prove useful in subsequent sections, may be calculated directly from this definition.

$$\begin{aligned}&\mathbb{E}_G[\beta^a_t] = 0 \qquad \mathbb{E}_G[\beta^a_t \beta^b_t] = \mathrm{e}^{ab} t \\ &\mathbb{E}_G[\beta^a_{t_1} \beta^b_{t_2}] = \mathrm{e}^{ab} \min(t_1, t_2) \\ &\mathbb{E}_G[(\beta^a_{t_2} - \beta^a_{t_1})(\beta^b_{t_2} - \beta^b_{t_1})] = \mathrm{e}^{ab} |t_2 - t_1|.\end{aligned} \tag{25}$$

An important consequence of these results is that the process β_t has *independent increments*:

$$\mathbb{E}_G[(\beta^a_{t_2} - \beta^a_{t_1})(\beta^b_{s_2} - \beta^b_{s_1})] = 0 \tag{26}$$

if $t_2 > t_1 \geqslant s_2 > s_1$.

These results show that anticommuting Brownian motion has the same covariance as the Itô Clifford process introduced by Barnett *et al* [13, 14] and further studied by Streater and Hasagawa [15]. From this point of view we are providing a concrete model of these processes, and applying them in a novel way to path integration in ghost quantum mechanics.

The results (25) can be further extended if we introduce the notion of adapted process in close analogy with the standard definition.

Definition 4.2. *A stochastic process F_t, $t \in [0, T]$ on m-dimensional anticommuting Wiener space such that for each t in $[0, T]$ F_t is a function of $\{\beta_s | 0 \leqslant s \leqslant t\}$ is said to be* $[0, t]$-adapted.

(The time interval, $[0, t]$, may be omitted when the context makes it clear.) As in the classical case, it can then be shown by direct calculation that, if F_t is a $[0, t]$-adapted process and $0 \leqslant s < u \leqslant T$, then

$$\begin{aligned}&\mathbb{E}_G[F^a_s(\beta^b_u - \beta^b_s)] = 0 \\ &\mathbb{E}_G[F^a_s(\beta^b_u - \beta^b_s)(\beta^c_u - \beta^c_s)] = \mathbb{E}_G[F^a_s] e^{bc}(u - s).\end{aligned} \tag{27}$$

5. Anticommuting stochastic integrals

As in the classical case, two kinds of integral of anticommuting stochastic processes will be useful, those with respect to time and those along (anticommuting) Brownian paths. Before defining these integrals it is useful to introduce a notation for a decreasing sequence of partitions of the interval $[0, t]$, $t \leqslant T$. For $N = 1, 2, \ldots$ and fixed t in $[0, T]$ the set $\{t^{[N]}_0, t^{[N]}_1, \ldots, t^{[N]}_N\}$ is a subset of $[0, T]$ with $t^{[N]}_0 = 0$, $t^{[N]}_0 < \cdots < t^{[N]}_N$, $t^{[N]}_N = t$ and $\Delta t^{[N]} \equiv \sup_{r=1\ldots N} |t^{[N]}_r - t^{[N]}_{r-1}| \to 0$ as $N \to \infty$.

Definition 5.1. *The* integral with respect to time *of an n-dimensional adapted process A^i_s is defined (when it exists independent of the choice of decreasing sequence of partitions) to be the process*

$$\int_0^t \mathrm{d}s\, A^i_s := \lim_{N\to\infty} \sum_{r=1}^{N} (t^{[N]}_r - t^{[N]}_{r-1}) A^i_{t^{[N]}_{r-1}}. \tag{28}$$

It is clearly $[0, t]$-adapted.

The anticommuting analogue of the Itô integral will now be defined.

Definition 5.2. *Suppose that C^i_{as}, $i = 1, \dots, n$, $a = 1, \dots, m$ is an $(n \times m)$-dimensional adapted process on anticommuting Wiener space. Then the* Itô integral *of the process is defined (when it exists independent of the choice of sequence of decreasing partitions) to be*

$$\int_0^t \mathrm{d}\beta^a_s \, C^i_{as} := \lim_{N\to\infty} \sum_{r=1}^{N} (\beta^a_{t_r} - \beta^a_{t_{r-1}}) C^i_{at_{r-1}}. \tag{29}$$

It is clearly $[0, t]$*-adapted.*

At this stage we do not consider necessary or sufficient conditions on the processes A_t, C_t for these integrals to exist; this question is addressed directly for the various processes considered in applications in later sections.

Definition 5.3. *An* anticommuting Itô process *or* anticommuting stochastic integral *is a process of the form*

$$Z^i_t = Z^i_0 + \int_0^t \mathrm{d}s \, A^i_s + \int_0^t \mathrm{d}\beta^a_s \, C^i_{as} \tag{30}$$

where A^i_s and C^i_{as} are $[0, s]$*-adapted processes.*

Using (25) and (27) the anticommuting Itô isometry can be proved in close analogy to the classical case [18].

Proposition 5.4. *Suppose that for $i = 1, \dots, k$*

$$Z^i_t = \int_0^t \mathrm{d}\beta^a_s C^i_{as}$$

with each Z^i of definite Grassmann parity. Then

$$\mathbb{E}_G[Z^i_t Z^j_t] = \int_0^t \mathrm{d}s \, \mathbb{E}_G[(-1)^{\epsilon_{Z^i}} \mathrm{e}^{ba} C^i_{as} C^j_{bs}]. \tag{31}$$

6. Anticommuting stochastic differential equations

In this section the anticommuting analogues of stochastic differential equations will be considered; these are applied in the final section to give a proof of the Feynman–Kac formula for a wide class of Hamiltonians. No very general theory is needed; a rather prescriptive and constructive approach is taken, motivated by the application to path integration. We simply define a sequence of random variables which satisfy the required stochastic differential equation.

Theorem 6.1. *Suppose that for $i = 1, \dots, n$ and $a = 1, \dots, m$ the functions A^i and C^i_a are supersmooth functions on $\mathbb{R}^{0,m}_S$ Suppose also that ζ_0 is an element of $\mathbb{R}^{0,m}_S$. Then there exists a unique adapted process ζ_t which satisfies the n-dimensional system of anticommuting stochastic differential equations*

$$\zeta^i_t =_\mu \zeta^i_0 + \int_0^t \mathrm{d}s \, A^i_s(\zeta_s) + \int_0^t \mathrm{d}\beta^a_s \, C^i_{as}(\zeta_s). \tag{32}$$

Outline of proof. To prove existence we construct a solution as the limit of an inductive process. Let the sequence $\zeta_{t,k}, k = 1, 2, \ldots, t \in [0, T]$ of n-dimensional anticommuting stochastic processes be defined by

$$\begin{aligned}\zeta^i_{t,0} &= \zeta^i_0\\ \zeta^i_{t,k+1} &= \zeta^i_0 + \int_0^t \mathrm{d}s\, A^i_s(\zeta_{s,k}) + \int_0^t \mathrm{d}\beta^a_s\, C^i_{as}(\zeta_{s,k}).\end{aligned} \tag{33}$$

Then, using the Itô isometry proposition 5.4, it may be proved by induction that there exists a positive constant A such that (for any pair of finite subsets $\{t_1, \ldots, t_r\}$, $\{t'_1, \ldots, t'_p\}$ of $[0, T]$ and corresponding pair of finite sets of multi-indices $\mu^{[1]}, \ldots, \mu^{[r]}, \nu^{[1]}, \ldots, \nu^{[p]}$)

$$\begin{aligned}|\mathbb{E}_G((\zeta_{t_1,k} - \zeta_{t_1,k-1})^{\mu^{[1]}} \ldots (\zeta_{t_r,k} - \zeta_{t_r,k-1})^{\mu^{[r]}} \zeta^{\nu^{[1]}}_{t'_{1,k-1}} \ldots \zeta^{\nu^{[p]}}_{t'_{p,k-1}})|_G\\ \leqslant \frac{(A^{|\mu^{[1]}|}t)^k}{k!} \cdots \frac{(A^{|\mu^{[r]}|}t)^k}{k!} (A^{|\nu^{[1]}|+\cdots+|\nu^{[p]}|})^{k-1}.\end{aligned} \tag{34}$$

This result may be used to show that for each t in $[0, T]$ and each μ in M_n the sequence $|\mathbb{E}_G(\zeta^\mu_{t,k})|_G$ is Cauchy and hence that $\zeta_{t,k}$ converges to an anticommuting random variable ζ_t satisfying (32).

To prove uniqueness, we suppose that ω_t is also a solution to (32). Then, again by induction over k, it can be shown that there exists a positive constant B such that $f_{t,k} := \sup_{\mu,\nu\in M_n,\mu\neq\emptyset} |\mathbb{E}_G(\omega^\mu_t - \zeta^\mu_{t,k})\zeta^\nu_{t,k}|_G$ satisfies

$$0 \leqslant f_{t,k} \leqslant B \int_0^t \mathrm{d}s f_{s,k}. \tag{35}$$

and hence that, for each t in $[0, T]$, $\lim_{k\to\infty} f_{t,k} = 0$, so that $\omega_t =_\mu \zeta_t$.

The stochastic differential equation (32) is often written in differential form as

$$\mathrm{d}\zeta^i_t = \mathrm{d}t\, A^i_s(\zeta_t) + \mathrm{d}\beta^a_t\, C^i_{at}(\zeta_t). \tag{36}$$

In order to exploit solutions to anticommuting stochastic differential equations to gain information about diffusions, the following Itô formula for stochastic integrals is essential. As in the classical Itô theorem, there is a second-order term which would not be present in the deterministic setting.

Theorem 6.2. *Let $X^i_t, i = 1, \ldots, p + q$ be a stochastic process on anticommuting Wiener space with X^i even for $i = 1, \ldots, p$ and X^i odd for $i = p + 1, \ldots, p + q$, and with each X^i having the form*

$$X^i_t = X^i_0 + \int_0^t \mathrm{d}s\, A^i(s, \zeta_s) + \int_0^t \mathrm{d}\beta^a_s C^i_a(s, \zeta_s) \tag{37}$$

where $\zeta^j, j = 1, \ldots, n'$ are solutions to an n'-dimensional system of anticommuting stochastic differential equations, β_t is m-dimensional anticommuting Brownian motion and the functions A^i, C^i_a such that there exists a positive constant K for which $|A^i(t, \cdot)|_G < K, |C^i_a(t, \cdot)|_G < K$ each t in $[0, T]$. Then, if F is a supersmooth function of p even and q odd variables (in the sense that $F(X^i) = \sum_{\mu\in M_q} F_\mu(X^1, \ldots, X^p)X^{\mu_1+p} \ldots X^{\mu_{|\mu|}+p}$ with each F_μ a smooth function of p even variables which, together with its first and second and third derivatives, is uniformly bounded) then

$$F(X_t)d =_\mu F(X_0) + \int_0^t \mathrm{d}X^i_s \partial_i F(X_s) + \tfrac{1}{2} \int_0^t \mathrm{d}s\, (-1)^{\epsilon_{X^i}} \mathrm{e}^{ab} C^i_b(X_s) C^j_a(X_s) \partial_j \partial_i F(X_s). \tag{38}$$

Outline of proof. For each of the sequence of decreasing partitions of $[0, t]$ we note that

$$F(X_t) - F(X_0) = \sum_{r=1}^{N} \Delta F_r \tag{39}$$

where $\Delta F_r = F(X_{t_r^{[N]}}) - F(X_{t_{r-1}^{[N]}})$. Now at the Nth approximation to the stochastic integrals X_t^i we have

$$\Delta F_r = \Delta X_r^i \partial_i F(X_{t_{r-1}^{[N]}}) + \tfrac{1}{2}\Delta X_r^j \Delta X_r^i \partial_i \partial_j F(X_{t_{r-1}^{[N]}}) + \text{higher-order terms} \tag{40}$$

where

$$\Delta X_r^i = A^i(t_{r-1}^{[N]}, \zeta_{t_{r-1}^{[N]}})\delta t_r^{[N]} + \delta\beta^a_{t_r^{[N]}} C_a^i(t_{r-1}^{[N]}, \zeta_{t_{r-1}^{[N]}}). \tag{41}$$

If we now take the kth approximation to ζ_t we can show by induction, using the anticommuting Itô isometry, that the only terms in the sum (39) which are of order $(\delta t_r^{[N]})^1$ are $\Delta X_t^i \partial_i F(X_{t_r^{[N]}})$ (coming from the first-order terms in the Taylor expansion) and $\frac{1}{2}\delta t_r^{[N]}(-1)^{\epsilon_{X^i}} \mathrm{e}^{ab} C_b^i(t_{r-1}^{[N]}, \zeta_{t_{r-1}^{[N]}}) C_a^j(t_{r-1}^{[N]}, \zeta_{t_{r-1}^{[N]}}) \partial_j \partial_i F(X_{t_r^{[N]}})$ from the second-order term. All other terms are of higher order in $\delta t_r^{[N]}$ and thus do not contribute to the sum in the limit as N tends to infinity.

A simple but useful special case of this theorem is the integration by parts formula contained in the following corollary.

Corollary 6.3. *The differential of the product of two stochastic integrals of the form (37) is given by the integration by parts formula*

$$\mathrm{d}(X_t^1 X_t^2) = X_t^1\, \mathrm{d}X_t^2 + \mathrm{d}X_t^1 X_t^2 + \tfrac{1}{2}(-1)^{\epsilon_{X^1}} \mathrm{e}^{ab} C_a^1(t, \zeta_t) C_b^2(t, \zeta_t)\, \mathrm{d}t. \tag{42}$$

An example of the solution of a particular stochastic differential equation will now be described; the process which solves the equation is the anticommuting analogue of the Ornstein–Uhlenbeck process.

Example 6.4. Consider the two-dimensional system of anticommuting stochastic differential equations

$$\zeta_t^i = \int_0^t \mathrm{d}s(-r\zeta_s^i) + \int_0^t \mathrm{d}\beta_s^a\, c_a^i \tag{43}$$

where $i, a = 1, 2$ and r, c_a^i are even constants. This may be solved using the same method as in the standard theory of stochastic calculus, by applying the anticommuting form of the Itô integration by parts formula to the product $\mathrm{e}^{rt}\zeta_t^i$, obtaining

$$\begin{aligned} \mathrm{e}^{rt}\zeta_t^j &= \zeta_0^j + \int_0^t \mathrm{d}(\mathrm{e}^{rs})\, \zeta_s^j + \int_0^t \mathrm{d}\zeta_t^i\, \mathrm{e}^{rs} \\ &= \zeta_0^j + r\mathrm{e}^{rs} \int_0^t \mathrm{d}s\, \zeta_s^j + \int_0^t \mathrm{d}s\, (-r\zeta^i \mathrm{e}^{rs}) + \int_0^t \mathrm{d}\beta_s^a\, c_a^i \mathrm{e}^{rs} \\ &= \zeta_0^j + \int_0^t \mathrm{d}\beta_s^a\, c_a^i \mathrm{e}^{rs} \end{aligned} \tag{44}$$

so that

$$\zeta_t^i = \zeta_0^i \mathrm{e}^{-rt} + \mathrm{e}^{-rt} \int_0^t \mathrm{d}\beta_s^a\, c_a^i \mathrm{e}^{rs}. \tag{45}$$

7. The anticommuting Feynman–Kac formula

In this section we prove a Feynman–Kac formula for Hamiltonians which are even, second-order differential operators on the space $\mathcal{F}(n)$ of supersmooth functions of n anticommuting variables of the form

$$H = \tfrac{1}{2} g^{kj} \partial_j \partial_k + \mathrm{i}\alpha^j \partial_j + v \tag{46}$$

where v is an even function in $\mathcal{F}(n)$, α^i, $i = 1, \ldots, n$ are odd functions and $g^{kj} = e^{ab} c_b^k c_a^j$ with c_b^k, $k = 1, \ldots, n$, $b = 1, \ldots, m$ even functions. The approach taken is similar to that used for conventional, commuting diffusions, as presented for instance in the books of Arnold [19], Friedman [20] and Øksendal [18].

Theorem 7.1. *If H is a Hamiltonian of the form (46) and t is in $[0, T]$ then for any F in $\mathcal{F}(n)$*

$$(\mathrm{e}^{-Ht} F)(\xi) = \mathbb{E}_G[\mathrm{e}^{-\int_0^t \mathrm{d}s\, v(\zeta_s)} F(\zeta_t)] \tag{47}$$

where ζ_t is the anticommuting diffusion which starts from ξ and satisfies

$$\mathrm{d}\zeta_t^j = -\mathrm{i}\, \mathrm{d}t\, \alpha^j(\zeta_t) + \mathrm{d}\beta_t^a\, c_a^j(\zeta_t). \tag{48}$$

Proof. For $t \in [0, T]$ define the operator U_t on $\mathcal{F}(n)$ by

$$U_t F(\xi) = \mathbb{E}_G(\mathrm{e}^{-\int_0^t v(\zeta_s)\, \mathrm{d}s} F(\zeta_t)). \tag{49}$$

Then, using the Itô formula (38), we find that

$$U_t F(\xi) - F(\xi) = \int_0^t \mathrm{d}s\, U_s H F(\xi) \tag{50}$$

so that $U_t = \exp -Ht$ as required. □

The first example of the application of this formula that we will consider gives the basic path integral formula for the flat Hamiltonian.

Example 7.2. Consider the Hamiltonian

$$H = \partial_1 \partial_2 \tag{51}$$

acting on $\mathcal{F}(2)$. Working on two-dimensional anticommuting Wiener space, the corresponding diffusion is the solution to

$$\mathrm{d}\zeta_t^a = \mathrm{d}\beta_t^a \qquad a = 1, 2 \tag{52}$$

starting from ξ. This has solution $\zeta_t = \xi + \beta_t$ so that

$$\begin{aligned} \mathrm{e}^{-Ht} F(\xi) &= \mathbb{E}_G[F(\xi + \beta_t)] \\ &= \int \mathrm{d}^2\eta\, t \exp\left(\frac{\eta^1 \eta^2}{t}\right) F(\xi + \eta) \\ &= \int \mathrm{d}^2\eta\, t \exp\left(\frac{(\eta^1 - \xi^1)(\eta^2 - \xi^2)}{t}\right) F(\eta) \end{aligned} \tag{53}$$

simply reflecting the fact that anticommuting Brownian motion is built from the heat kernel of this very Hamiltonian.

A closely related example gives the basic path integral formula for the flat Hamiltonian with potential.

Example 7.3. For the Hamiltonian

$$H = \partial_1\partial_2 + v \tag{54}$$

(with v an even function) acting on $\mathcal{F}(2)$

$$\mathrm{e}^{-Ht}F(\xi) = \mathbb{E}_G(\mathrm{e}^{-\int_0^t \mathrm{d}s\, v(\xi+\beta_s)}(F(\xi+\beta_t))). \tag{55}$$

The next example, which is also two dimensional, concerns the Hamiltonian whose heat kernel gives the distribution for the anticommuting Ornstein–Uhlenbeck process described in example 6.4.

Example 7.4. In the case of the Hamiltonian

$$H = c^2\partial_1\partial_2 + r(\eta^1\partial_1 + \eta^2\partial_2) \tag{56}$$

we must consider the diffusion ζ_t starting from ξ and satisfying

$$\mathrm{d}\zeta_t^a = -r\zeta_t^a\,\mathrm{d}t + c\,\mathrm{d}\beta_t^a \tag{57}$$

so that using (45)

$$\zeta_t^a = \xi^a\mathrm{e}^{-rt} + \mathrm{e}^{-rt}\int_0^t \mathrm{d}\beta_s^a\,\mathrm{e}^{rs}. \tag{58}$$

Applying the Feynman–Kac formula to four functions which form a basis of $\mathcal{F}(2)$, that is, $F_0(\eta) = 1$, $F_1(\eta) = \eta^1$, $F_2(\eta) = \eta^2$ and $F_{12}(\eta) = \eta^1\eta^2$, we obtain

$$\begin{aligned}
&\exp^{-Ht} F_0(\xi) = \mathbb{E}_G[1] = 1\\
&\exp^{-Ht} F_1(\xi) = \mathbb{E}_G\left[\xi^1\mathrm{e}^{-rt} + \mathrm{e}^{-rt}\int_0^t c\,\mathrm{d}\beta_s^1\,\mathrm{e}^{rs}\right] = \mathrm{e}^{-rt}\xi^1\\
&\exp^{-Ht} F_2(\xi) = \mathrm{e}^{-rt}\xi^2\\
&\exp^{-Ht} F_{12}(\xi) = \mathbb{E}_G\left[\left(\xi^1\mathrm{e}^{-rt} + \mathrm{e}^{-rt}\int_0^t c\,\mathrm{d}\beta_s^1\,\mathrm{e}^{rs}\right)\left(\xi^2\mathrm{e}^{-rt} + \mathrm{e}^{-rt}\int_0^t c\,\mathrm{d}\beta_s^2\,\mathrm{e}^{rs}\right)\right]\\
&\qquad = \mathrm{e}^{-2rt}\xi^1\xi^2 + \int_0^t \mathrm{d}s\,c^2\mathrm{e}^{2rs}\mathrm{e}^{-2rt} = \mathrm{e}^{-2rt}\xi^1\xi^2 + \frac{c^2}{2r}(1-\mathrm{e}^{-2rt})
\end{aligned} \tag{59}$$

so that the heat kernel for this Hamiltonian is

$$\mathrm{e}^{-Ht}(\xi,\eta) = \eta^1\eta^2 - \mathrm{e}^{-rt}(\xi^1\eta^2 + \eta^1\xi^2) + \frac{c^2}{2r}(1-\mathrm{e}^{-2rt}) + \mathrm{e}^{-2rt}\xi^1\xi^2. \tag{60}$$

The next example we consider is the anticommuting harmonic oscillator. This is the fundamental example in BRST quantization in the sense that quantizing a quantum mechanical system with k momenta constrained to be zero leads to a ghost Hamiltonian with the form of the $2k$-dimensional anticommuting harmonic oscillator [21, 22]. For simplicity we consider only the two-dimensional case.

Example 7.5. Consider the Hamiltonian

$$H = \partial_1\partial_2 - \eta^1\eta^2 \tag{61}$$

which leads to the anticommuting diffusion

$$\zeta_t^a = \xi^a + \beta_t^a \qquad a = 1, 2. \tag{62}$$

The anticommuting Feynman–Kac formula for this diffusion is

$$(\mathrm{e}^{-Ht}F)(\xi) = \mathbb{E}_G[\mathrm{e}^{\int_0^t \mathrm{d}s(\xi^1+\beta_s^1)(\xi^2+\beta_s^2)}F(\xi+\beta_t)]. \tag{63}$$

To evaluate this integral for finite t we will use essentially the same technique as that employed by Simon in [23]. To achieve this we need to extract the kernel of the time evolution operator from this expression, and define the analogue of conditional expectation. Taking the definition of the expectation with respect to anticommuting Brownian motion (63) becomes (we put $\Delta t := t_r - t_{r-1}$)

$$(\mathrm{e}^{-Ht}F)(\eta) = \lim_{N\to\infty}\int_{\mathcal{B}} \mathrm{d}^2\eta_1\cdots\mathrm{d}^2\eta_N\, p(\eta_1,\Delta t)p(\eta_2-\eta_1,\Delta t)\cdots p(\eta_N-\eta_{N-1},\Delta t)$$
$$\times\exp\left(\sum_{r=0}^{N-1}\Delta t(\eta^1+\eta_r^1)(\eta^2+\eta_r^2)\right)F(\eta+\eta_N). \tag{64}$$

Making a change of variables $\eta_r \mapsto \eta_r' := \eta + \eta_r$, dropping the primes and replacing η_N by η', we obtain

$$(\mathrm{e}^{-Ht}F)(\eta) = \int_{\mathcal{B}} \mathrm{d}\eta'\,(\mathrm{e}^{-Ht}F)(\eta,\eta')F(\eta') \tag{65}$$

where

$$\begin{aligned}
&(\mathrm{e}^{-Ht}F)(\eta,\eta')\\
&\quad= \lim_{N\to\infty}\int_{\mathcal{B}} \mathrm{d}\eta_1\cdots\mathrm{d}\eta_{N-1}\, p(\eta_1-\eta,\Delta t)p(\eta_2-\eta_1,\Delta t)\cdots p(\eta_{N-1}-\eta_{N-2},\Delta t)\\
&\qquad\times p(\eta'-\eta_{N-1},\Delta t)\exp\left(\sum_{r=0}^{N-1}\Delta t\,\eta_r^1\eta_r^2\right)\\
&\quad= \lim_{N\to\infty}\int_{\mathcal{B}} \mathrm{d}\eta_1\cdots\mathrm{d}\eta_{N-1}\, p(\eta_1-\eta,\Delta t)p(\eta_2-\eta_1,\Delta t)\cdots p(\eta_{N-1}-\eta_{N-2},\Delta t)\\
&\qquad\times p(\eta'-\eta_{N-1},\Delta t)\frac{p(\eta-\eta',t)}{p(\eta-\eta',t)}\exp\left(\sum_{r=0}^{N-1}\Delta t\,\eta_r^1\eta_r^2\right)\\
&\quad:= p(\eta-\eta',t)\mathbb{E}_G\left[\exp\left(\int_0^t \mathrm{d}s\,\omega_s^1\omega_s^2\right)\middle|\,\omega_0=\eta,\omega_t=\eta'\right]
\end{aligned} \tag{66}$$

defining both the process ω_t (which will be called *pinned anticommuting Brownian motion*), and the conditional expectation operator.

Following Simon, we use the Brownian bridge to represent such pinned Brownian motion processes. The (two-dimensional) *anticommuting Brownian bridge* process starting and ending at 0, over the time interval [0, 1] is defined by

$$\alpha_s^i := \beta_s^i - s\beta_1^i. \tag{67}$$

In close analogy with the classical case, it may be confirmed using (25) that this process has covariance

$$\mathbb{E}_G[\alpha_s^i\alpha_u^j] = \mathrm{e}^{ij}s(1-u) \tag{68}$$

for $0 \leqslant s \leqslant u \leqslant 1$. This allows us to express ω_t as

$$\omega^i(s) = \eta^i\left(1-\frac{s}{t}\right)+\eta'^i\frac{s}{t}+t^{1/2}\alpha^i\left(\frac{s}{t}\right). \tag{69}$$

Since $\int_0^t \mathrm{d}s\, f(s/t) = t\int_0^1 \mathrm{d}s'\, f(s')$, we can restrict our attention to ω_s for $0 \leqslant s \leqslant 1$.

We now take the Fourier expansion of $\alpha(s)$,

$$\alpha_s^i = \sum_{r=1}^{\infty}\ell_r\xi_r^i f_r(s) \qquad i = 1, 2 \tag{70}$$

where $\ell_r := (r\pi)^{-1}$, $f_r(s) := \sqrt{2}\sin(r\pi s)$ and the ξ_r are the anticommuting analogue of independent Gaussian random variables, that is to say, their formal measure is

$$\prod_{r=1}^{\infty}(\mathrm{d}^2\xi_r\ \exp\xi_r^1\xi_r^2). \tag{71}$$

It can be confirmed (as in the book of Simon [23] for the classical case) that this Fourier expansion for the Brownian bridge gives the same covariance as (68) above when expectations are taken using this formal measure.

Pinned Brownian motion $\omega(s)$ thus has the Fourier expansion

$$\omega^i(s) = \sum_{r=1}^{\infty} f_r(s)(\gamma_r^i + \sqrt{t}\ell_r\xi_r^i) \tag{72}$$

where $\gamma_r^i = \sqrt{2}\ell_r(\eta^i + (-1)^{r+1}\eta'^i)$. Substituting this into the expression (66) for the kernel of the time evolution operator we obtain

$$\begin{aligned}
(\mathrm{e}^{-Ht}F)(\eta,\eta') &= p(\eta-\eta',t)\int_{\mathcal{B}}\left(\prod_{r=1}^{\infty}\mathrm{d}^2\xi_r\exp\xi_r^1\xi_r^2\right)\\
&\quad\times\exp\left[\int_0^1\mathrm{d}s\,t\left(\sum_{r=1}^{\infty}f_r(s)(\gamma_r^1+\sqrt{t}\ell_r\xi_r^1)\right)\left(\sum_{r=1}^{\infty}f_r(s)(\gamma_r^2+\sqrt{t}\ell_r\xi_r^2)\right)\right]\\
&= p(\eta-\eta',t)\int_{\mathcal{B}}\left(\prod_{r=1}^{\infty}\mathrm{d}\xi_r\exp\xi_r^1\xi_r^2\right)\exp\sum_{r=1}^{\infty}t(\gamma_r^1+\sqrt{t}\ell_r\xi_r^1)(\gamma_r^2+\sqrt{t}\ell_r\xi_r^2)\\
&= p(\eta-\eta',t)\int_{\mathcal{B}}\left(\prod_{r=1}^{\infty}\mathrm{d}\xi_r\exp\xi_r^1\xi_r^2\right)\\
&\quad\times\exp\sum_{r=1}^{\infty}t^2\ell_r^2(\gamma_r^1t^{-1/2}\ell_r^{-1}+\xi_r^1)(\gamma_r^2t^{-1/2}\ell_r^{-1}+\xi_r^2).
\end{aligned} \tag{73}$$

Evaluating the Gaussian integrals we obtain

$$\begin{aligned}
(\mathrm{e}^{-Ht}F)(\eta,\eta') &= t\prod_{r=1}^{\infty}(1+t^2\ell_r^2)\exp\left[(\eta^1\eta^2+\eta'^1\eta'2)\left(\frac{1}{t}+\sum_{r=1}^{\infty}\frac{2t\ell_r^2}{1+t^2\ell_r^2}\right)\right]\\
&\quad\times\exp\left[(\eta^1\eta'^2+\eta'^1\eta^2)\left(\frac{1}{t}+\sum_{r=1}^{\infty}\frac{2(-1)^rt\ell_r^2}{1+t^2\ell_r^2}\right)\right].
\end{aligned} \tag{74}$$

Using the Weierstrass–Hadamard factorization of $\sinh x$:

$$\sinh x = x\prod_{r=1}^{\infty}(1+\ell_r^2x^2) \tag{75}$$

and the Mittag–Leffler expansions of $(\sinh x)^{-1}$ and $\coth x$:

$$(\sinh x)^{-1} = \frac{1}{x}+\sum_{r=1}^{\infty}\frac{2(-1)^rx\ell_r^2}{1+x^2\ell_r^2} \qquad \coth x = \frac{1}{x}+\sum_{r=1}^{\infty}\frac{2x\ell_r^2}{1+x^2\ell_r^2} \tag{76}$$

we finally find the kernel for the time evolution operator to be

$$(\mathrm{e}^{-Ht}F)(\eta,\eta') = \sinh t\ \exp\left[\frac{1}{\sinh t}[(\eta^1\eta^2+\eta'^1\eta'^2)\cosh t-(\eta^1\eta'^2+\eta'^1\eta^2)]\right]. \tag{77}$$

Finally we consider an example with quartic fermionic terms.

Example 7.6. Consider the Hamiltonian

$$H = (c^2 + 2b\eta^1\eta^2)\frac{\partial^2}{\partial\eta^2\partial\eta^1}. \tag{78}$$

Following theorem 7.1 we consider the stochastic differential equation

$$\zeta_t^a = \xi^a + \int_0^t \mathrm{d}\beta_s^a \left(a + \frac{b}{a}\zeta_s^1\zeta_s^2\right). \tag{79}$$

Without actually solving this equation it can be seen by direct calculation (together with the anticommuting Itô isometry proposition 5.4) that

$$\begin{aligned} &\mathbb{E}_G[1] = 1 \qquad \mathbb{E}_G[\zeta_t^a] = \xi^a \quad a = 1, 2 \\ &\mathbb{E}_G[\zeta_t^1\zeta_t^2] = \xi^1\xi^2\mathrm{e}^{-2bt} + \frac{c^2}{2b}(\mathrm{e}^{-2bt} - 1) \end{aligned} \tag{80}$$

giving the action of e^{-HT} on the four elementary functions 1, η^1, η^2 and $\eta^1\eta^2$ to be

$$\begin{aligned} &\exp -Ht[1] = 1 \qquad \mathbb{E}_G[\eta_t^a] = \eta^a \quad a = 1, 2 \\ &\mathbb{E}_G[\eta_t^1\eta_t^2] = \eta^1\eta^2\mathrm{e}^{-2bt} + \frac{c^2}{2b}(\mathrm{e}^{-2bt} - 1) \end{aligned} \tag{81}$$

leading to the expression of the heat kernel as

$$\mathrm{e}^{-Ht}(\eta, \xi) = \delta(\eta - \xi) + \frac{c^2}{2b}(\mathrm{e}^{-2bt} - 1). \tag{82}$$

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